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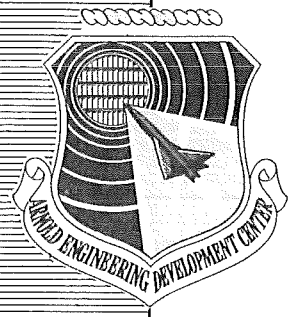

TRANSFORMATION OF AXES SYSTEMS BY MATRIX METHODS AND APPLICATION TO WIND TUNNEL DATA REDUCTION

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ARNOLD ENGINEERING DEVELOPMENT CENTER
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TRANSFORMATION OF AXES SYSTEMS BY MATRIX METHODS AND APPLICATION TO WIND TUNNEL DATA REDUCTION

By

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von Kármán Gas Dynamics Facility

ARO, Inc.

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ABSTRACT

A method is described which simplifies the derivation of many wind tunnel data reduction equations. Standard matrix techniques are used for the solution of the simultaneous equations encountered in the transformation of vector components from one axes system to another. Two typical applications are presented: the determination of aerodynamic angles and the transfer of aerodynamic loads from body to wind axes.

PUBLICATION REVIEW

This report has been reviewed and publication is approved.

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NOMENCLATURE

A	Arbitrary vector
A_{Xn}, A_{Yn}, A_{Zn}	Vector components of A directed along the X_n-, Y_n-, and Z_n-axes, respectively
F_A	Axial force, body axes
F_C	Crosswind force, wind axes
F_D	Drag force, wind axes
F_L	Lift force, wind axes
F_N	Normal force, body axes
F_Y	Side force, body axes
M_X	Rolling moment, body axes
M_{Xw}	Rolling moment, wind axes
M_Y	Pitching moment, body axes
M_{Yw}	Pitching moment, wind axes
M_Z	Yawing moment, body axes
M_{Zw}	Yawing moment, wind axes
$[M]$	Transformation matrix, subscripts indicate rotation angles and order of rotation
$[M^{-1}]$	Inverse of transformation matrix
$[M']$	Transpose of transformation matrix
$\|M\|$	Determinant of transformation matrix
u, v, w	Velocity components directed along the X_n-, Y_n-, and Z_n-axes, respectively
V	Free-stream velocity
X_n, Y_n, Z_n	Orthogonal axes, subscript indicates number of rotational operations performed on axes system
α	Angle of attack, angle between the projection of the wind X-axis on the body X, Z-plane and the body X-axis
α_i	Indicated pitch angle
α_p	Sting prebend angle in the body X, Z-plane

β	Angle of sideslip, angle between the wind X-axis and the projection of this axis on the body X, Z-plane
θ	Pitch angle, angle of rotation about the Y-axis, positive when the +Z-axis is rotated into the +X-axis
ϕ	Roll angle, angle of rotation about the X-axis, positive when the +Y-axis is rotated into the +Z-axis
ϕ_i	Indicated roll angle
ψ	Yaw angle, angle of rotation about the Z-axis, positive when the +X-axis is rotated into the +Y-axis

1.0 INTRODUCTION

In the development of wind tunnel data reduction programs, it is frequently necessary to transform vectors, such as forces and velocity components, from one rectangular Cartesian coordinate system to another. The transformation may be accomplished by rotating the axes system through a succession of angles until the desired axes system is obtained. The relation between the reference-axes system and the transformed-axes system may then be determined by solving the set of simultaneous equations which result from the individual rotations. The solution of these equations is greatly simplified by the use of matrix techniques. In the following sections, transformation matrices are developed for rotation about each of the orthogonal axes and then the method of combining a series of rotations is described. Typical applications are given to demonstrate the method.

2.0 DEVELOPMENT OF THE TRANSFORMATION MATRICES

An orthogonal right-hand axes system is used in the development of the transformation matrices. The three basic orientation angles are defined in Refs. 1 and 2 as: The roll angle, ϕ, which results from a rotation of the axes system about the X-axis; the pitch angle, θ, which results from a rotation about the Y-axis; and the yaw angle, ψ, which results from a rotation about the Z-axis. The positive directions of the axes are indicated in Fig. 1, and the rotation about an axis is defined as positive when it appears clockwise to an observer looking along the axis in the positive direction. The transformation matrices derived in this section are correct only if these conventions are observed.

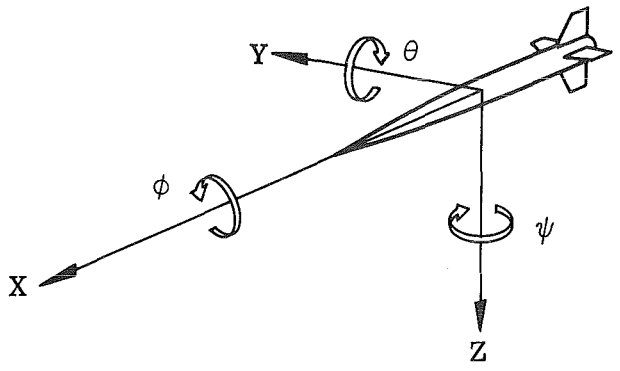

Fig. 1 Axes System and Orientation Angle Notation

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Basic transformation matrices are developed first for each of the rotation angles.

2.1 ROLL TRANSFORMATION MATRIX

Consider an arbitrary vector A whose magnitude, by definition, is unchanged by axis transformations. Resolve the vector into a system of three orthogonal components A_{X0}, A_{Y0}, and A_{Z0} parallel to the X_0-, Y_0-, and Z_0-axes, respectively. If the axes system is rotated through an angle ϕ about the X_0-axis, a set of transformed vector components A_{X1}, A_{Y1}, and A_{Z1} are obtained as shown in Fig. 2.

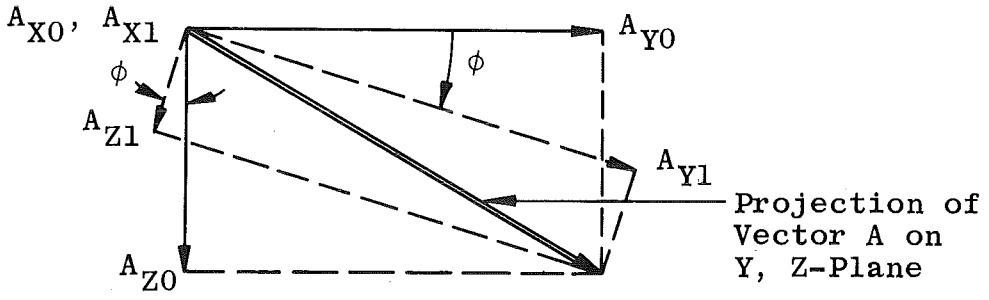

Fig. 2 Roll Rotation, Viewed in the +X Direction

The relation between the transformed vector components and the original vector components is given by

$$\begin{aligned} A_{X1} &= (1)\,A_{X0} + (0)\,A_{Y0} + (0)\,A_{Z0} \\ A_{Y1} &= (0)\,A_{X0} + (\cos\phi)\,A_{Y0} + (\sin\phi)\,A_{Z0} \\ A_{Z1} &= (0)\,A_{X0} + (-\sin\phi)\,A_{Y0} + (\cos\phi)\,A_{Z0} \end{aligned} \tag{1}$$

Writing Eq. (1) in matrix form,

$$\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = [M_\phi] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{2}$$

where the roll transformation matrix $[M_\phi]$ is given by

$$[M_\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{3}$$

2.2 PITCH TRANSFORMATION MATRIX

Consider the system of vector components A_{X0}, A_{Y0}, and A_{Z0} to be rotated through an angle θ about the Y_0-axis. This gives the transformed vector components A_{X1}, A_{Y1}, and A_{Z1} as shown in Fig. 3.

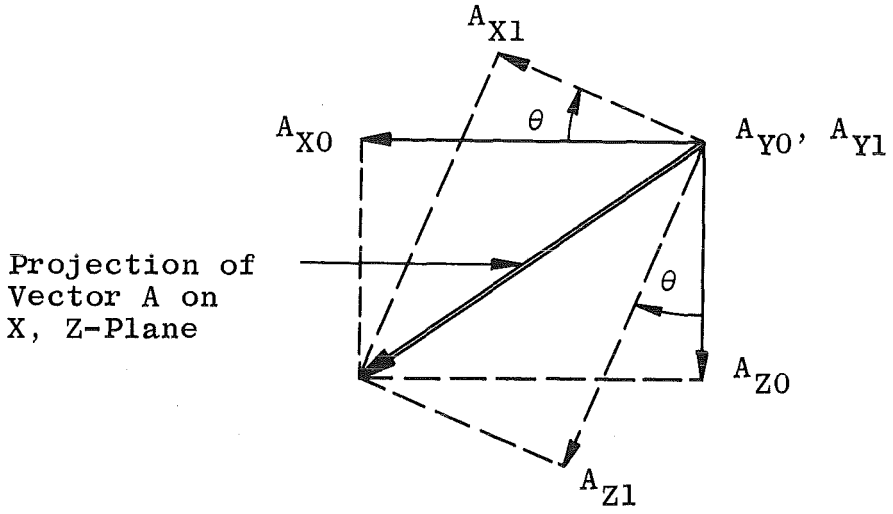

Fig. 3 Pitch Rotation, Viewed in the +Y Direction

The equations of the transformed vector components are

$$\begin{aligned} A_{X1} &= (\cos\theta)A_{X0} + (0)A_{Y0} + (-\sin\theta)A_{Z0} \\ A_{Y1} &= (0)A_{X0} + (1)A_{Y0} + (0)A_{Z0} \\ A_{Z1} &= (\sin\theta)A_{X0} + (0)A_{Y0} + (\cos\theta)A_{Z0} \end{aligned} \tag{4}$$

Writing Eq. (4) in matrix form,

$$\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = [M\theta] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{5}$$

where the pitch transformation matrix $[M\theta]$ is given by

$$[M\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{6}$$

2.3 YAW TRANSFORMATION MATRIX

Consider the system of vector components A_{X0}, A_{Y0}, and A_{Z0} to be rotated through an angle ψ about the Z_0-axis. This gives the transformed vector components A_{X1}, A_{Y1}, and A_{Z1} as shown in Fig. 4.

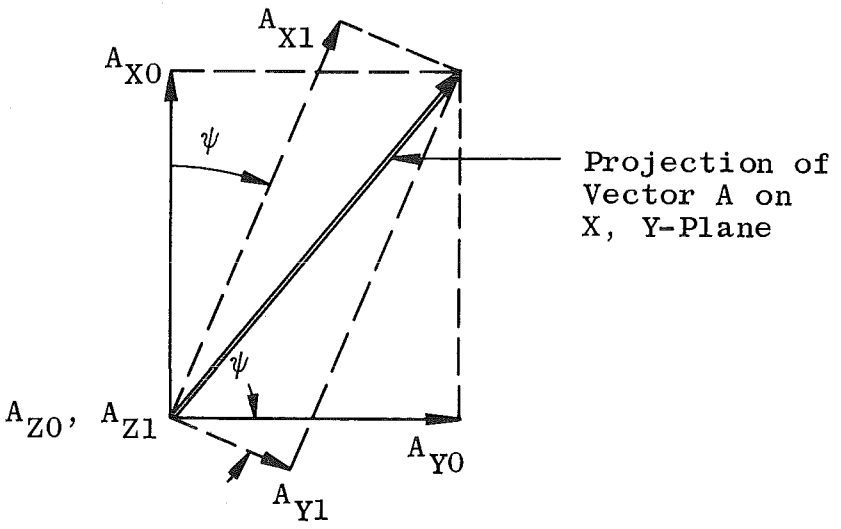

Fig. 4 Yaw Rotation, Viewed in the +Z Direction

The equations of the transformed vector components are

$$\begin{aligned} A_{X1} &= (\cos\psi)\,A_{X0} + (\sin\psi)\,A_{Y0} + (0)\,A_{Z0} \\ A_{Y1} &= (-\sin\psi)\,A_{X0} + (\cos\psi)\,A_{Y0} + (0)\,A_{Z0} \\ A_{Z1} &= (0)\,A_{X0} + (0)\,A_{Y0} + (1)\,A_{Z0} \end{aligned} \tag{7}$$

Writing Eq. (7) in matrix form,

$$\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = [M_\psi] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{8}$$

where the yaw transformation matrix $[M_\psi]$ is given by

$$[M_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

2.4 MULTIPLE OPERATIONS

In most aerodynamic applications, more than one rotation of the axes system is required and two or more of the preceding operations must be combined. Assume that a set of vector components A_{X0}, A_{Y0}, and A_{Z0} are transformed to a new axes system by letting the initial axes system be rolled about the X_0-axis, pitched about the Y_1-axis, and yawed about the Z_2-axis in that order. The transformation equations for each rotation are given by Eqs. (2), (5), and (8). Then,

$$\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = [M_\phi] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} = [M_\theta] \begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} \tag{10b}$$

$$\begin{bmatrix} A_{X3} \\ A_{Y3} \\ A_{Z3} \end{bmatrix} = [M_\psi] \begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} \tag{10c}$$

Substitution of Eqs. (10a) and (10b) into Eq. (10c) will eliminate the intermediate axes 1 and 2. Thus, the transformed vector components A_{X3}, A_{Y3}, and A_{Z3} are obtained directly in terms of the initial vector components:

$$\begin{bmatrix} A_{X3} \\ A_{Y3} \\ A_{Z3} \end{bmatrix} = [M_\psi][M_\theta][M_\phi] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \equiv [M_{\psi\theta\phi}] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{11}$$

where the transformation matrix $[M_{\psi\theta\phi}]$ is the product of the three basic transformation matrices. Assuming that $[M_{\psi\theta\phi}]$ has the form

$$[M_{\psi\theta\phi}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{12}$$

then Eq. (11) can be written as

$$\begin{aligned} A_{X3} &= a_{11} A_{X0} + a_{12} A_{Y0} + a_{13} A_{Z0} \\ A_{Y3} &= a_{21} A_{X0} + a_{22} A_{Y0} + a_{23} A_{Z0} \\ A_{Z3} &= a_{31} A_{X0} + a_{32} A_{Y0} + a_{33} A_{Z0} \end{aligned} \tag{13}$$

In combining the basic matrices to obtain the product $[M_{\psi\theta\phi}]$, the proper order must be maintained since the transformation matrices are not commutative. Therefore, $[M_\theta]$ must operate on $[M_\phi]$ and their product $[M_{\theta\phi}]$ is operated on by the matrix $[M_\psi]$. The mechanics of matrix multiplication are given in many texts (e.g., Ref. 3), and an example should be sufficient here. Let

$$[M_\theta] = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \tag{14a}$$

and

$$[M_\phi] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{14b}$$

To obtain the element in the i th row and the j th column of the product $[M_{\theta\phi}]$, form a sum of the products of the elements of the i th row

of $[M_\theta]$ and the corresponding elements of the j th column of $[M_\phi]$. Thus, operating on $[M_\phi]$ with $[M_\theta]$ gives

$$[M_\theta][M_\phi] \equiv [M_{\theta\phi}] = \begin{bmatrix} Aa + Bd + Cg & Ab + Be + Ch & Ac + Bf + Ci \\ Da + Ed + Fg & Db + Ee + Fh & Dc + Ef + Fi \\ Ga + Hd + Ig & Gb + He + Ih & Gc + Hf + Ii \end{bmatrix} \tag{15}$$

A similar operation is used to obtain

$$[M_{\psi\theta\phi}] = [M_\psi][M_{\theta\phi}] \tag{16}$$

An unlimited number of rotations may be used with this method to obtain a set of transformed axes. The specific applications will define the number of rotations required and their sequence. However, it is essential that the order of combining the transformation matrices is kept in the proper sequence as follows:

$$\begin{bmatrix} A_{Xn} \\ A_{Yn} \\ A_{Zn} \end{bmatrix} = [M_{n\text{ th rotation}}] \ldots [M_2][M_1] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{17}$$

where $[M_2]$ operates on $[M_1]$, and $[M_3]$ operates on their product $[M_{21}]$, etc.

The elements of a transformation matrix denote the direction cosines of the transformed axes system referred to the fixed axes system. For example, in Eq. (12) the element a_{11} is the direction cosine of the A_{X3}-axis referred to the A_{X0}-axis. This property is of use in many applications and also provides a check on the transformation matrix since the determinant of the matrix, $|M_{\psi\theta\phi}|$, must equal unity.

It is often useful to have the initial vector components given as explicit functions of the transformed vector components. This relation may be obtained by using the inverse matrix $[M^{-1}\psi\theta\phi]$ since the transformation matrix is a non-singular square matrix. Multiplying both sides of Eq. (11) by the inverse matrix gives

$$\begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} = [M^{-1}\psi\theta\phi] \begin{bmatrix} A_{X3} \\ A_{Y3} \\ A_{Z3} \end{bmatrix} \tag{18}$$

In general, the inverse matrix is difficult to evaluate. However, since the transformation matrices consist of the direction cosines of three mutually perpendicular axes, they are orthogonal matrices and have the characteristic that the inverse matrix is equal to the transpose of the matrix, $[M'\psi\theta\phi]$. The transpose is easily determined since the i th row of the transpose is just the i th column of the matrix. Thus, from Eq. (12),

$$[M^{-1}\psi\theta\phi] = [M'\psi\theta\phi] = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{19}$$

Then, from Eqs. (18) and (19),

$$\begin{aligned} A_{X0} &= a_{11} A_{X3} + a_{21} A_{Y3} + a_{31} A_{Z3} \\ A_{Y0} &= a_{12} A_{X3} + a_{22} A_{Y3} + a_{32} A_{Z3} \\ A_{Z0} &= a_{13} A_{X3} + a_{23} A_{Y3} + a_{33} A_{Z3} \end{aligned} \tag{20}$$

An application of the inverse of a transformation matrix is given in Section 3.2.2.

3.0 TYPICAL APPLICATIONS

Since matrix multiplication is not commutative in general, it is essential that the correct sequence of rotation is used in applying the previously derived matrices. For example, the transformed-axes system obtained by pitching and then rolling an axes system is not coincident with the axes system obtained by rolling and then pitching the same initial system. Also, in applying the basic transformation matrices, the previously defined sign conventions must be followed.

Two typical applications of transformation matrix techniques to wind tunnel data reduction problems are presented to demonstrate the method.

3.1 DETERMINATION OF MODEL ATTITUDE

In the analysis of wind tunnel data it is usually necessary to determine the model angles of attack and sideslip as functions of the indicated wind tunnel angles.

If the free-stream velocity vector V is resolved into three orthogonal components u, v, and w along the body X-, Y-, and Z-axes, respectively, the aerodynamic angles are defined (e.g., Refs. 1 and 2) by the relations

$$\alpha = \tan^{-1} w/u \tag{21}$$

$$\beta = \sin^{-1} v/V \tag{22}$$

These angles are shown in Fig. 5.

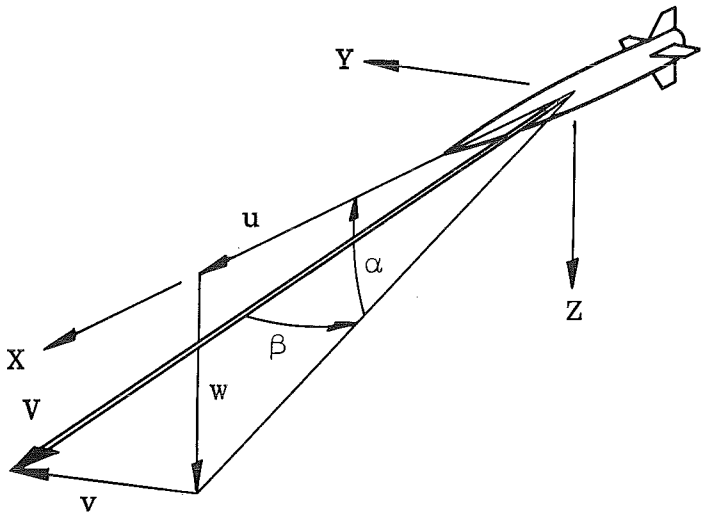

Fig. 5 Velocity Components and Aerodynamic Angles

A common procedure used in supersonic and hypersonic wind tunnel testing for placing a model at combined angles of attack and sideslip is to use a mechanism which pitches the model through an indicated angle α_i in the tunnel X, Z-plane and then rolls the model through an angle ϕ_i about the body X-axis. A bent sting, which pitches the model through an angle α_p in the body X, Z-plane, may be used in conjunction with the pitch mechanism to obtain larger angles of attack.

To determine the relation between the free-stream velocity V and the body-axes velocity components, three orthogonal vector components A_{X0}, A_{Y0}, and A_{Z0} in the tunnel axes are transformed to the body axes by the following rotation sequence: (1) The axes system is pitched through an angle α_i about the tunnel Y_0-axis, (2) the system is then rolled through an angle ϕ_i about the X_1-axis, and (3) the system is then pitched through

an angle α_p about the Y_2-axis. The transformed vector components A_{X3}, A_{Y3}, and A_{Z3} in the body axes are given by

$$\begin{bmatrix} A_{X3} \\ A_{Y3} \\ A_{Z3} \end{bmatrix} = [M_{\alpha_p}][M_{\phi_i}][M_{\alpha_i}] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \equiv [M_{\alpha_p\ \phi_i\ \alpha_i}] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{23}$$

From Eqs. (3) and (6),

$$[M_{\alpha_i}] = \begin{bmatrix} \cos\alpha_i & 0 & -\sin\alpha_i \\ 0 & 1 & 0 \\ \sin\alpha_i & 0 & \cos\alpha_i \end{bmatrix} \tag{24a}$$

$$[M_{\phi_i}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_i & \sin\phi_i \\ 0 & -\sin\phi_i & \cos\phi_i \end{bmatrix} \tag{24b}$$

$$[M_{\alpha_p}] = \begin{bmatrix} \cos\alpha_p & 0 & -\sin\alpha_p \\ 0 & 1 & 0 \\ \sin\alpha_p & 0 & \cos\alpha_p \end{bmatrix} \tag{24c}$$

Then,

$$[M_{\alpha_p\ \phi_i\ \alpha_i}] = \begin{bmatrix} \cos\alpha_p \cos\alpha_i & \sin\alpha_p \sin\phi_i & -\cos\alpha_p \sin\alpha_i \\ -\sin\alpha_p \cos\phi_i \sin\alpha_i & & -\sin\alpha_p \cos\phi_i \cos\alpha_i \\ \sin\phi_i \sin\alpha_i & \cos\phi_i & \sin\phi_i \cos\alpha_i \\ \sin\alpha_p \cos\alpha_i & -\cos\alpha_p \sin\phi_i & -\sin\alpha_p \sin\alpha_i \\ +\cos\alpha_p \cos\phi_i \sin\alpha_i & & +\cos\alpha_p \cos\phi_i \cos\alpha_i \end{bmatrix} \tag{25}$$

From Eq. (23),

$$\begin{aligned} A_{X3} &= A_{X0}(\cos\alpha_p \cos\alpha_i - \sin\alpha_p \cos\phi_i \sin\alpha_i) + A_{Y0}(\sin\alpha_p \sin\phi_i) \\ &\quad - A_{Z0}(\cos\alpha_p \sin\alpha_i + \sin\alpha_p \cos\phi_i \cos\alpha_i) \\ A_{Y3} &= A_{X0}(\sin\phi_i \sin\alpha_i) + A_{Y0}(\cos\phi_i) + A_{Z0}(\sin\phi_i \cos\alpha_i) \\ A_{Z3} &= A_{X0}(\sin\alpha_p \cos\alpha_i + \cos\alpha_p \cos\phi_i \sin\alpha_i) - A_{Y0}(\cos\alpha_p \sin\phi_i) \\ &\quad - A_{Z0}(\sin\alpha_p \sin\alpha_i - \cos\alpha_p \cos\phi_i \cos\alpha_i) \end{aligned} \tag{26}$$

Referring to Fig. 5, the orthogonal vector components can be replaced by velocity components. Thus,

$$\begin{aligned} A_{X0} &= V \qquad A_{X3} = u \\ A_{Y0} &= 0 \qquad A_{Y3} = v \\ A_{Z0} &= 0 \qquad A_{Z3} = w \end{aligned} \tag{27}$$

Then, from Eqs. (26) and (27),

$$\begin{aligned} u &= V(\cos\alpha_p \cos\alpha_i - \sin\alpha_p \cos\phi_i \sin\alpha_i) \\ v &= V \sin\phi_i \sin\alpha_i \\ w &= V(\sin\alpha_p \cos\alpha_i + \cos\alpha_p \cos\phi_i \sin\alpha_i) \end{aligned} \tag{28}$$

The angle of attack, α, is given by Eqs. (21) and 28):

$$\alpha = \tan^{-1}\frac{w}{u} = \tan^{-1}\left(\frac{\sin\alpha_p \cos\alpha_i + \cos\alpha_p \cos\phi_i \sin\alpha_i}{\cos\alpha_p \cos\alpha_i - \sin\alpha_p \cos\phi_i \sin\alpha_i}\right) \tag{29a}$$

which reduces to

$$\alpha = \alpha_p + \tan^{-1}(\tan\alpha_i \cos\phi_i) \tag{29b}$$

and from Eqs. (22) and (28) the sideslip angle, β, is

$$\beta = \sin^{-1}\frac{v}{V} = \sin^{-1}(\sin\alpha_i \sin\phi_i) \tag{30}$$

3.2 TRANSFER OF AERODYNAMIC FORCES AND MOMENTS

With the general use of internal balances in wind tunnel testing, data are usually obtained in the body axes. Thus, the transformation of body-axes forces and moments to other axes systems is often required. Two examples of the application of transformation matrices to force and moment transfer are presented.

3.2.1 Body to Wind Axes

The orientation of three orthogonal vector components (A_{X0}, A_{Y0}, A_{Z0}) in the body axes and their relationship to the transformed vector components in the wind axes (A_{X2}, A_{Y2}, A_{Z2}) is shown in Fig. 6.

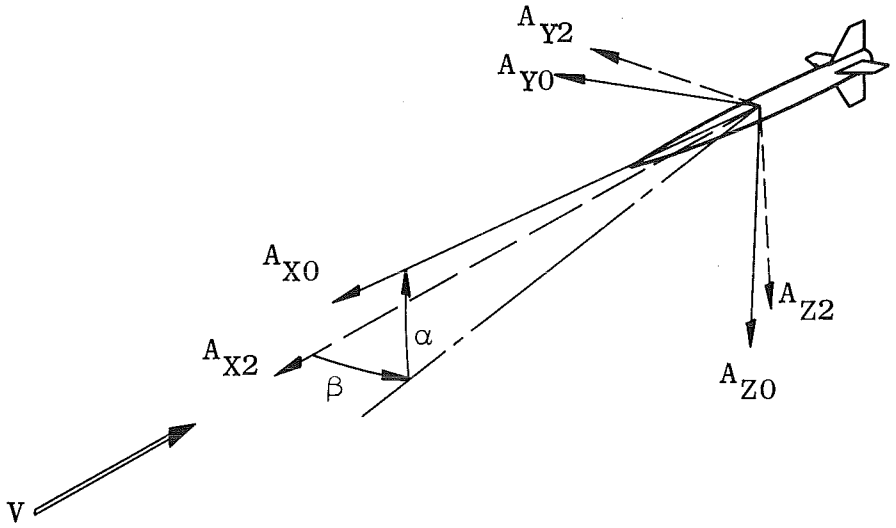

Fig. 6 Vector Components in Body- and Wind-Axes Systems

Transformation of the body-axes components to the wind axes is accomplished by the following sequence of rotations: (1) pitch through an angle $-\alpha$ about the Y_0-axis, and (2) yaw through an angle β about the Z_1-axis. Then,

$$\begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} = [M_\beta][M_{-\alpha}] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \equiv [M(\beta)(-\alpha)] \begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} \tag{31}$$

From Eqs. (6) and (9), the rotation matrices are

$$[M_{-\alpha}] = \begin{bmatrix} \cos(-\alpha) & 0 & -\sin(-\alpha) \\ 0 & 1 & 0 \\ \sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} \tag{32a}$$

$$[M_\beta] = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{32b}$$

Combining these matrices in the proper order gives

$$\left[M(\beta)(-\alpha)\right] = \begin{bmatrix} \cos\beta\cos\alpha & \sin\beta & \cos\beta\sin\alpha \\ -\sin\beta\cos\alpha & \cos\beta & -\sin\beta\sin\alpha \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{33}$$

Then Eq. (31) gives

$$\begin{aligned} A_{X2} &= A_{X0}\cos\beta\cos\alpha + A_{Y0}\sin\beta + A_{Z0}\cos\beta\sin\alpha \\ A_{Y2} &= A_{X0}(-\sin\beta\cos\alpha) + A_{Y0}\cos\beta + A_{Z0}(-\sin\beta\sin\alpha) \\ A_{Z2} &= A_{X0}(-\sin\alpha) + A_{Y0}(0) + A_{Z0}\cos\alpha \end{aligned} \tag{34}$$

By definition, the aerodynamic forces are related to the vector components as follows:

$$\begin{aligned} A_{X0} &= -F_A & A_{X2} &= -F_D \\ A_{Y0} &= F_Y & A_{Y2} &= F_C \\ A_{Z0} &= -F_N & A_{Z2} &= -F_L \end{aligned} \tag{35}$$

Substituting these relations in Eq. (34) provides the results

$$\begin{aligned} F_D &= F_A\cos\beta\cos\alpha - F_Y\sin\beta + F_N\cos\beta\sin\alpha \\ F_C &= F_A\sin\beta\cos\alpha + F_Y\cos\beta + F_N\sin\beta\sin\alpha \\ F_L &= -F_A\sin\alpha + F_N\cos\alpha \end{aligned} \tag{36}$$

Transfer of moments can be accomplished by replacing each moment by its equivalent vector component. Then,

$$\begin{aligned} A_{X0} &= M_X & A_{X2} &= M_{Xw} \\ A_{Y0} &= M_Y & A_{Y2} &= M_{Yw} \\ A_{Z0} &= M_Z & A_{Z2} &= M_{Zw} \end{aligned} \tag{37}$$

and from Eq. (34),

$$\begin{aligned} M_{Xw} &= M_X\cos\beta\cos\alpha + M_Y\sin\beta + M_Z\cos\beta\sin\alpha \\ M_{Yw} &= -M_X\sin\beta\cos\alpha + M_Y\cos\beta - M_Z\sin\beta\sin\alpha \\ M_{Zw} &= -M_X\sin\alpha + M_Z\cos\alpha \end{aligned} \tag{38}$$

3.2.2 Wind to Body Axes

To obtain the body-axes components in terms of the wind-axes components, the inverse of Eq. (33) is used.

$$\left[M^{-1}_{(\beta)(-\alpha)}\right] = \begin{bmatrix} \cos\beta\cos\alpha & -\sin\beta\cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \cos\beta\sin\alpha & -\sin\beta\sin\alpha & \cos\alpha \end{bmatrix} \tag{39}$$

then

$$\begin{bmatrix} A_{X0} \\ A_{Y0} \\ A_{Z0} \end{bmatrix} = \left[M^{-1}_{(\beta)(-\alpha)}\right] \begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} \tag{40}$$

and

$$\begin{aligned} A_{X0} &= A_{X2}\cos\beta\cos\alpha + A_{Y2}(-\sin\beta\cos\alpha) + A_{Z2}(-\sin\alpha) \\ A_{Y0} &= A_{X2}\sin\beta + A_{Y2}\cos\beta + A_{Z2}(0) \\ A_{Z0} &= A_{X2}\cos\beta\sin\alpha + A_{Y2}(-\sin\beta\sin\alpha) + A_{Z2}\cos\alpha \end{aligned} \tag{41}$$

Thus, the body-axes forces and moments are

$$\begin{aligned} F_A &= F_D\cos\beta\cos\alpha + F_C\sin\beta\cos\alpha - F_L\sin\alpha \\ F_Y &= -F_D\sin\beta + F_C\cos\beta \\ F_N &= F_D\cos\beta\sin\alpha + F_C\sin\beta\sin\alpha + F_L\cos\alpha \\ M_X &= M_{X_W}\cos\beta\cos\alpha - M_{Y_W}\sin\beta\cos\alpha - M_{Z_W}\sin\alpha \\ M_Y &= M_{X_W}\sin\beta + M_{Y_W}\cos\beta \\ M_Z &= M_{X_W}\cos\beta\sin\alpha - M_{Y_W}\sin\beta\sin\alpha + M_{Z_W}\cos\alpha \end{aligned} \tag{42}$$

3.3 OTHER APPLICATIONS

The derivation of weight tare corrections on balance loads can be handled in a manner similar to that described in Section 3.1. For an equivalent series of rotations, Eq. (26) can be applied directly with the appropriate choice of the initial and final vector components. Weight tare corrections for arbitrarily oriented balances such as those used in instrumented controls are easily handled by additional rotations.

A procedure similar to that used in Section 3.2 can be used for transformation of loads in any axes system to any other axes system by the appropriate application of the transformation matrices.

4.0 CONCLUDING REMARKS

A method has been developed which simplifies the transformation of vector components from one rectangular Cartesian coordinate system to another. Matrix techniques are utilized to solve the simultaneous equations resulting from the rotation of an axes system through the three basic orientation angles. The number of axes rotations and their sequence is not limited.

The method has been found to be very useful in the development of wind tunnel data reduction programs.

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